

# Scale invariance and superfluid turbulence

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## ABSTRACT

We construct a Schroedinger field theory invariant under local spatial scaling. It is shown to provide an effective theory of superfluid turbulence by deriving, analytically, the observed Kolmogorov 5/3 law and to lead to a Biot-Savart interaction between the observed filament excitations of the system as well.

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# 1 Introduction

We describe a construction of an  $1 + 3$ -dimensional Schroedinger field theory which is invariant under local scaling in the three spatial dimensions. This is effected by introducing a gauge field and a spatial metric. The requirement of local scaling in three dimensions allows for a Chern Simons term in the action but forbids a Maxwell term. The locally scale invariant action is unique in the sense that it contains all possible terms having polynomial interaction among the Schroedinger field, the gauge field and the metric. Moreover, gauge invariance for this system is rather novel due to the presence of the metric.

Historically, local scale invariance was introduced [1] in an attempt to unifying the theories of gravitation and electromagnetism. Stipulating local scale invariance of the theory of General Relativity in four dimensions lead to the extremely novel idea of introducing a gauge field with an additional term in the action resembling Maxwell's theory. Identifying this term as electromagnetism a unification of the theories of gravitation and electromagnetism was deemed to be have been achieved through purely geometric means. This approach was criticised, however, as being incompatible with the observed discrete spectra of atoms [2]. The idea was thus given up as a means to producing a unified field theory only to be revived later on with the local scaling of lengths replaced by a local change of phase of a quantum wave function [3]. This construction is now known as "gauge theory" although it is no longer related to length scales. What is retained, however, is the idea of introducing a gauge field in order to render a system invariant under a local symmetry.

In this article we consider a spatially scale invariant generalization of the Schroedinger field theory in the original sense of Weyl. Unlike the original approach, however, gauge variations, that is local changes of scale, are compensated for by a Ricci term rather than by a Maxwell term. The theory constructed would be relevant for describing any three-dimensional non-relativistic quantum field theory which has scale invariance properties. The system constructed is not meant to be a microscopic theory but an effective theory for any locally scale invariant theory of this type. A good testing ground for this effective theory is the phenomenon of turbulence in superfluid helium.

Turbulence in superfluid liquid Helium [4] exhibits Kolmogorov scaling [5] and does not have any associated discrete spectrum. Thus it could be a good testing ground for our effective theory. The usual theoretical approaches to superfluid turbulence uses the non-linear Gross-Pitaevski (GP) equation [6] where the nonlinearity reflects the interaction between helium atoms in the field theory description of the system taking a superfluid condensate into account. When energy is injected into the system, say by heating, excitations in the form of filaments appear. In this approach the observed filament excitations are understood with their location given by the zeros of the GP wave function. The filament excitations can also be modelled more directly with their dynamics described in analogy with interaction of wires carrying currents obeying the Biot-Savart law [7].

Distribution functions in superfluid turbulence are different from those arising for classical fluids. For instance, the velocity distribution function is not Gaussian but has a power law tail [8]. We find that the unique locally scale invariant theory constructed here contains the appropriate degrees of freedom for describing superfluid turbulence namely, a condensate and the filament

excitations. Indeed, we show using standard arguments of weak wave turbulence that the theory constructed predicts the Kolmogorov 5/3 law observed in a certain range of momenta of the quasi-particle excitations of the theory. While numerical studies of the GP equation yield similar results [9], the present analysis is completely analytic.

The model presented can accommodate both filaments and condensates which appear in all approaches used to describe superfluid turbulence. The locus of the zeros of the condensate field gives the location of the filaments, which are modelled as currents coupled to the gauge field supported only on the filaments. An effective theory for the Schroedinger field is obtained by integrating out the gauge field from the theory. This produces the action for a GP-like system, while the effective theory for the filaments is that of Wilson lines interacting through the Chern Simons term. This leads to a Biot-Savart type of interaction between the filaments. This interaction can be used to estimate the velocity of separation between a pair of filaments that collide. The result we obtain, valid for small times, is in agreement with the experimental results [5].

The plan of the article is as follows. In section 2 we derive the unique scale invariant action starting from the action of a free Schroedinger field. A solution to the equations of motion for a special arrangement of filaments is presented in section 3. In section 4 we obtain effective theories for the Schroedinger field and Wilson lines before concluding in section 5.

## 2 Scale invariant action

In this section we construct an action invariant under spatial scaling starting from the action for a free Schroedinger field. The gauge group is  $\mathbf{R}^*$ , the group of non-zero reals, which is non-compact. The free Schroedinger equation, in operator form, allows Bose-Einstein condensation and is thus an appropriate starting point for a theory of quantum turbulence. First, the free system is made invariant under global scaling by introducing a time-independent metric for the three spatial directions. It is then made invariant under local scaling by introducing a gauge field.

The action of the Schroedinger field  $\psi$  in  $\mathbf{R}^1 \times \mathbf{R}^3$ , with the first factor designating time,  $t$ , and the second one corresponding to the spatial coordinates  $\mathbf{x} = (x^1, x^2, x^3) = (x, y, z)$  is

$$\mathcal{S}(\psi, g) = i \int \psi^* \partial_t \psi \sqrt{g} dt d^3x - \frac{1}{2m} \int g^{ij} \partial_i \psi^* \partial_j \psi \sqrt{g} dt d^3x, \quad (1)$$

where we have introduced a metric  $g$  on  $\mathbf{R}^3$  and  $\partial_i$  denotes the derivative with respect to  $x^i$  and an asterisk designates complex conjugation. The second term of the action is invariant under the global scaling transformation of the field  $\psi$  and the metric

$$\begin{aligned} \psi &\longmapsto e^{-\Lambda/4} \psi, \\ g_{ij} &\longmapsto e^{\Lambda} g_{ij}, \end{aligned} \quad (2)$$

where  $\Lambda$  is a constant. Let us note that the scale invariance could not be effected without the metric. Moreover, as mentioned before, we do not impose scale invariance on the first term involving temporal derivative of the Schroedinger field. We now promote this global scaling symmetry to

a local symmetry by allowing spatial dependence of  $\Lambda$  [10] and introducing a gauge field  $A_i$  and define covariant derivatives of the field  $\psi$  and the metric  $g$  as [11]

$$\begin{aligned} D_i \psi &= \partial_i \psi - \alpha A_i \psi, \\ D_i g_{km} &= \partial_i g_{km} + 4\alpha A_i g_{km}, \end{aligned} \quad (3)$$

where  $\alpha$  is a real parameter. It appears from (3) that the parameter  $\alpha$  may be dispensed with by a redefinition of the gauge field. However, the sign of  $\alpha$  is of import in obtaining field configurations and will be fixed later. Then under the gauge transformation

$$\begin{aligned} \psi &\longmapsto e^{-\Lambda(\mathbf{x})/4} \psi, \\ g_{ij} &\longmapsto e^{\Lambda(\mathbf{x})} g_{ij}, \\ A_i &\longmapsto A_i - \frac{1}{4\alpha} \partial_i \Lambda(\mathbf{x}), \end{aligned} \quad (4)$$

with space-dependent  $\Lambda$ , the covariant derivatives of the scalar field  $\psi$  and the metric transform as

$$\begin{aligned} D_i \psi &\longmapsto e^{-\Lambda(\mathbf{x})/4} D_i \psi, \\ D_i g_{jk} &\longmapsto e^{\Lambda(\mathbf{x})} D_i g_{jk}. \end{aligned} \quad (5)$$

Hence replacing the derivatives with respect to the spatial coordinates in the second term of (1) by covariant derivatives we obtain the action

$$\mathcal{S}(\psi, A, g) = i \int \psi^* \partial_t \psi \sqrt{g} dt d^3x - \frac{1}{2m} \int g^{ij} D_i \psi^* D_j \psi \sqrt{g} dt d^3x, \quad (6)$$

which is invariant under the gauge transformations (4). One can add one more gauge-invariant term to the above action involving the curvature and the gauge field [12]. To this end let us define Christoffel symbols [11] as

$$\tilde{\Gamma}_{jk}^i = \frac{1}{2} g^{im} (D_j g_{mk} + D_k g_{mj} - D_m g_{jk}). \quad (7)$$

By (5), the Christoffel symbol is invariant under the local scaling transformations (4). Then the Ricci tensor ensuing from this Christoffel symbol defined as

$$\tilde{R}_{jkl}^i = \partial_l \tilde{\Gamma}_{jk}^i - \partial_k \tilde{\Gamma}_{jl}^i + \tilde{\Gamma}_{ml}^i \tilde{\Gamma}_{jk}^m - \tilde{\Gamma}_{mk}^i \tilde{\Gamma}_{jl}^m \quad (8)$$

is also invariant under the gauge transformation (4). The resulting scalar curvature defined as

$$\tilde{R} = g^{jl} \tilde{R}_{jl}^i \quad (9)$$

then transforms as  $\tilde{R} \longmapsto e^{-\Lambda} \tilde{R}$  under (4). Hence,

$$\int |\psi|^2 \tilde{R} \sqrt{g} dt d^3x \quad (10)$$

is invariant under the gauge transformation. It can be checked that no other term involving curvature tensors or derivatives of  $A$  or their combinations can be made gauge invariant in this fashion

to yield a local polynomial action. In particular, the term  $F_{ij}^2$  constructed from the gauge field  $A_i$  is not scale invariant in three dimensions, nor can it be made gauge invariant in a polynomial action.

The Ricci scalar  $\tilde{R}$  defined above can be related to the Ricci scalar corresponding to the metric  $g$  by expanding  $\tilde{\Gamma}_{jk}^I$  using (3) [11], resulting into

$$\begin{aligned}\tilde{R} &= R + 8\alpha\nabla_i A^i + 8\alpha^2 A^2, \\ &= R + \frac{8\alpha}{\sqrt{g}}\partial_i(\sqrt{g}A^i) + 8\alpha^2 A^2,\end{aligned}\tag{11}$$

where we used  $A^2 = g^{ij}A_iA_j = A^iA_i$ ,  $\nabla_i$  and  $R$  denote, respectively, the covariant derivative with respect to  $x^i$  and the scalar curvature corresponding to the metric  $g$ .

Putting (11) in (10) and adding to (6) along with a Chern Simons term for the gauge field we obtain the unique spatially scale-invariant action in 1 + 3 dimensions given by

$$\begin{aligned}\mathcal{S}(\psi, A, g) &= \int \sqrt{g}dt d^3x \left( i\psi^*\partial_t\psi - \frac{1}{2m}g^{ij}(\partial_i\psi^*\partial_j\psi - \alpha A_i\partial_j|\psi|^2 + \alpha^2 A_iA_j|\psi|^2) \right) \\ &+ \beta \int dt d^3x |\psi|^2 (\sqrt{g}R + 8\alpha\partial_i(\sqrt{g}A^i) + 8\alpha^2\sqrt{g}A^2) + \gamma \int dt d^3x \epsilon^{ijk} A_i\partial_j A_k,\end{aligned}\tag{12}$$

where  $\epsilon^{ijk}$  denotes the rank three antisymmetric tensor,  $\beta$  and  $\gamma$  are real parameters. The Chern-Simons term being independent of the metric is locally scale invariant. From the four-dimensional perspective, this term is to be thought of as the unique potential term for the gauge field which has local three-dimensional scale invariance. Furthermore, while it does not contribute to the equations of motion, this term plays a crucial role, as we shall see below, in determining the interaction between filaments modelled by the gauge field. We now proceed to study the properties of this unique locally scale invariant three-dimensional system.

### 3 Solution for special geometrical arrangement

Physical configurations are obtained as solutions to the Euler-Lagrange equations ensuing from the action (12) by variation of the fields. The equation obtained upon varying the metric  $g$  is

$$\begin{aligned}-\frac{1}{2}g_{ij} \left( i\psi^*\partial_t\psi - \frac{1}{2m}(\partial_i\psi^*\partial_j\psi - \alpha A_i\partial_j|\psi|^2 + \alpha^2 A_iA_j|\psi|^2) + \beta|\psi|^2 (R + 8\alpha\nabla_i A^i + 8\alpha^2 A^2) \right) \\ - \frac{1}{2m}(\partial_i\psi^*\partial_j\psi - \alpha A_i\partial_j|\psi|^2 + \alpha^2 A_iA_j|\psi|^2) + \beta|\psi|^2 (R_{ij} + 8\alpha\nabla_i A_j + 8\alpha^2 A_iA_j) = 0\end{aligned}\tag{13}$$

Equations arising from the variations of the gauge field  $A$  and the Schroedinger field  $\psi^*$  are, respectively,

$$\partial_i|\psi|^2 = 2\alpha A_i|\psi|^2,\tag{14}$$

$$i\sqrt{g}\partial_t\psi + \frac{1}{2m}\partial_i(\sqrt{g}g^{ij}\partial_j\psi) + \beta\sqrt{g} \left( R + \alpha(8 - \frac{1}{2m\beta})\nabla_i A^i + \alpha^2(8 - \frac{1}{2m\beta})A^2 \right) \psi = 0,\tag{15}$$

where we assumed

$$16m\beta \neq 1. \quad (16)$$

The Chern Simons term does not contribute to the equations of motion.

Let us focus on the equations of motion for stationary configurations,  $\partial_t \psi = 0$ . Assuming the Schroedinger field to be time-independent, equations (14) and (13) together lead to

$$R + 8\alpha \nabla_i A^i + \alpha^2 \left(8 + \frac{1}{2m\beta}\right) A^2 = \frac{1}{2m\beta} \frac{\partial_i \psi^* \partial^i \psi}{|\psi|^2}, \quad (17)$$

where in deriving this division by  $|\psi|^2$  was used, so that the equation is valid only for non-vanishing  $|\psi|^2$ . For stationary configurations thus we need only to consider equations (14), (17) and (15) with the first term set to zero..

Equation (14) is solved with

$$|\psi|^2 = |\psi_0|^2 \exp \left( 2\alpha \int_C A_i dx^i \right), \quad (18)$$

where  $\psi_0$  is a constant and  $C$  the curve over which the line integral is evaluated.

Filaments can be introduced in this model now by introducing currents  $J(\mathbf{x}) = J_{Ci} dx^i$  supported on a curve  $C$ , by delta functions,  $J_{Ci} = J_i^{(0)} \delta_C^{(3)}$ , where  $J_i^{(0)}$  is a constant. We thus have

$$\int_C A_i dx^i = \int_{\mathbf{R}^3} A_i J^i(C) d^3x, \quad (19)$$

leading to

$$|\psi|^2 = |\psi_0|^2 \exp \left( 2\alpha J^{(0)i} \int_{\mathbf{R}^3} A_i \delta_{(C)}^{(3)} d^3x \right). \quad (20)$$

We also relate the constant  $\psi_0$  to the number of filaments  $N$  as  $\psi_0 = \sqrt{N/V}$ ,  $V$  denoting the volume of the superfluid. Choosing the constant  $\alpha$  to be negative and  $J^{(0)i}$  to be positive by convention, the modulus of  $\psi$  vanishes on the curve and is a non-zero constant everywhere else, equal to  $|\psi_0|^2$ . The constant wave function can be interpreted as representing a Bose-Einstein condensate. The filaments then represent excitations of the system corresponding to injection of energy. Thus the simple model has features which suggest that excitations can be described either as filaments or as zeros of the condensate. Moreover, the metric is  $\eta_{ij}$  in the bulk of the condensate. In view of this in the next section we proceed to construct an effective action for the system in terms of the wave function by integrating out the gauge field and in terms of the gauge field by integrating out the wave function.

## 4 Effective theories

As mentioned in the introduction, two approaches for studying superfluid turbulence are either using a GP equation of a Bose-Einstein condensate or in terms of interacting vortex filaments.

In the previous section we found that the local scale invariant theory allows for both of these configurations. We now proceed to construct an effective theory for the Schroedinger field by integrating out the gauge field from the action (12). As mentioned earlier, the scope of such a unique scale invariant theory is rather vast. However, in view of the results of the previous section the metric pertaining to superfluid bulk is flat. Therefore, we set the metric to be the Euclidean one,  $g_{ij} = \eta_{ij}$  in the bulk of the condensate and define  $S(\psi, A) = S(\psi, A, \eta)$ . As excitations in the superfluid background are filaments located at the zeroes of the Schroedinger field, the effective theory for the filaments is given by Wilson lines in the background of a Chern Simons theory.

## 4.1 The condensate and quasi-particle spectrum

First let us integrate out the gauge field from the action (12) with a flat metric to obtain the effective action for the condensate  $\psi$  defined by the path integral

$$e^{iS_{\text{eff}}(\psi)} = \frac{1}{\sqrt{\pi}} \int \mathcal{D}A e^{iS(\psi, A, \eta)}. \quad (21)$$

Setting  $g_{ij} = \eta_{ij}$  in (12), we obtain, up to boundary terms

$$\begin{aligned} S(\psi, A, \eta) = & i \int \psi^* \partial_t \psi - \frac{1}{2m} \int \partial_i \psi^* \partial_i \psi - \frac{\hat{g}}{4} \int (\partial_i \log |\psi|^2)^2 |\psi|^2 \\ & + \hat{g} \int \left( A_i - \frac{1}{2} \partial_i \log |\psi|^2 \right)^2 |\psi|^2 + \gamma \int \epsilon^{ijk} A_i \partial_j A_k. \end{aligned} \quad (22)$$

where we defined  $\hat{g} = 8\alpha\beta - \frac{1}{2m}$  and suppressed the measure  $dt d^3x$  in the integrals. We now redefine the gauge field with a shift, namely,

$$\tilde{A}_i = A_i - \frac{1}{2} \partial_i \log |\psi|^2. \quad (23)$$

Then in the Chern-Simons term

$$\int \epsilon^{ijk} A_i \partial_j A_k = \int \epsilon^{ijk} \tilde{A}_i \partial_j \tilde{A}_k, \quad (24)$$

up to boundary terms. Integrating out with respect to the new field  $\tilde{A}$  we obtain the effective action

$$S_{\text{eff}}(\psi) = i \int \psi^* \partial_t \psi - \frac{1}{2m} \int \partial_i \psi^* \partial_i \psi - \frac{\hat{g}}{4} \int (\partial_i \log |\psi|^2)^2 |\psi|^2 + \gamma \int \epsilon^{ijk} \tilde{A}_i \partial_j \tilde{A}_k + \Gamma. \quad (25)$$

where the effective potential

$$\Gamma = -\frac{1}{2} \int_0^L \frac{d\xi}{\xi} \int d^3x e^{-\xi \hat{g} |\psi|^2}. \quad (26)$$

Expanding the exponential and performing the integration with respect to  $\xi$ , the effective potential becomes

$$\Gamma = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (\hat{g}L)^n}{n! n} \int |\psi|^{2n}, \quad (27)$$

where we have neglected an infinite constant term ensuing from the unit term in the exponential.

Let us now consider the quasi-particle spectrum of this theory [13, 14]. Considering stationary configurations we expand  $\psi$  in Fourier modes

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \psi^*(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad (28)$$

where  $a_{\mathbf{k}}^\dagger$  and  $a_{\mathbf{k}}$  are, respectively, creation and annihilation operators for the bosonic modes satisfying the commutation relation

$$[a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'} \quad (29)$$

The sum is over all momentum modes. For each momentum mode we define a number operator  $\hat{n}(k) = a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ , depending only on the magnitude of the momentum, thanks to the rotational symmetry. The states diagonalizing these number operators satisfy

$$\hat{n}(k)|n(\mathbf{k})\rangle = n(k)|n(\mathbf{k})\rangle, \quad a_{\mathbf{k}}|n(\mathbf{k})\rangle = \sqrt{n(k)}|n(\mathbf{k})-1\rangle, \quad a_{\mathbf{k}}^\dagger|n(\mathbf{k})\rangle = \sqrt{n(k)+1}|n(\mathbf{k})+1\rangle. \quad (30)$$

For the zero momentum mode we also assume the existence of a state  $|\psi_0\rangle = |n(0)\rangle$  with

$$a_0|\psi_0\rangle = a_0^\dagger|\psi_0\rangle = \sqrt{N}|\psi_0\rangle, \quad (31)$$

where we denoted  $n(0) = N$  and assumed  $N$  to be sufficiently large so that  $\sqrt{N} \sim \sqrt{N+1}$ . This state corresponds to the condensate over which the non-zero modes are taken to be fluctuations. Substituting (28) in (27) we obtain

$$\Gamma = -\frac{1}{2} \sum_{\substack{\mathbf{k}'_1, \mathbf{k}'_2, \dots, \mathbf{k}'_n \\ \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{g^n}{n} a_{\mathbf{k}'_1}^\dagger a_{\mathbf{k}'_2}^\dagger \cdots a_{\mathbf{k}'_n}^\dagger a_{\mathbf{k}_1} a_{\mathbf{k}_2} \cdots a_{\mathbf{k}_n} \delta(\mathbf{k}'_1 + \mathbf{k}'_2 + \cdots + \mathbf{k}'_n - \mathbf{k}_1 - \mathbf{k}_2 - \cdots - \mathbf{k}_n), \quad (32)$$

where we denoted  $g = \hat{g}L/V$ . So far we have not fixed the parameters. We now assume that  $g = 1/N$ . Then in  $\Gamma$  the quadratic terms  $a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ ,  $a_{-\mathbf{k}} a_{\mathbf{k}}$  and  $a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger$  arise with  $N^{n-1}$  in the  $n$ -th term, while all other terms are lower order in  $N$ . The effective potential becomes

$$\Gamma = -\frac{1}{2} \sum_{\mathbf{k}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{g}{n} \left( n^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \binom{n}{2} a_{-\mathbf{k}} a_{\mathbf{k}} + \binom{n}{2} a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger \right) + \mathcal{O}(1/N) \times \text{quartic terms}. \quad (33)$$

The coefficients of the quadratic terms are determined by the number of ways of satisfying the momentum conservation constraint

$$\mathbf{k}'_1 + \mathbf{k}'_2 + \cdots + \mathbf{k}'_n = \mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_n. \quad (34)$$

For example, the term  $a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$  is obtained as one chooses any one  $\mathbf{k}'$  as well as any single  $\mathbf{k}$  to be non-zero, which can be chosen in  $n \times n$  ways. The term  $a_{-\mathbf{k}} a_{\mathbf{k}}$  is obtained by choosing all  $\mathbf{k}'$  to be zero and two of the  $n$   $\mathbf{k}$ 's to be non-zero. The third term is obtained similarly.



Now, along with the kinetic term, the Hamiltonian reads, upon performing the sum over  $n$ ,

$$H = \sum_{\mathbf{k} \neq 0} \left( 2\ell_1 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - \ell_2 (a_{-\mathbf{k}} a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger) \right), \quad (35)$$

where we defined

$$\ell_1 = \frac{1}{2} \left( \frac{k^2}{2m} + \frac{g}{2e} \right), \quad \ell_2 = \frac{g}{2e} \left( \frac{e}{2} - 1 \right). \quad (36)$$

Let us note that the third term involving the derivative of  $\ln |\psi|^2$  comes with the coupling constant  $\hat{g} = gV/L$ , which can be ignored compared with  $g$ . In order to obtain the quasi-particle spectrum we need to diagonalize the Hamiltonian. To this end we change basis as

$$a_{\mathbf{k}} = u\alpha_{\mathbf{k}} + v\alpha_{-\mathbf{k}}^\dagger \quad (37)$$

$$a_{\mathbf{k}}^\dagger = u\alpha_{\mathbf{k}}^\dagger + v\alpha_{-\mathbf{k}}, \quad (38)$$

where  $u$  and  $v$  are taken to be real parameters. Requiring the commutations relations

$$[\alpha_{\mathbf{k}}^\dagger, \alpha_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}, \quad (39)$$

in addition to (29) for any momentum  $\mathbf{k}$ , we obtain the constraint  $u^2 - v^2 = 1$ , so that the two bases are related by a Bogoliubov transformation

$$a_{\mathbf{k}} = \alpha_{\mathbf{k}} \cosh \theta + \alpha_{-\mathbf{k}}^\dagger \sinh \theta \quad (40)$$

$$a_{\mathbf{k}}^\dagger = \alpha_{\mathbf{k}}^\dagger \cosh \theta + \alpha_{-\mathbf{k}} \sinh \theta, \quad (41)$$

where  $\theta$  is a real parameter. Then expressing the Hamiltonian in terms of the new oscillators  $\alpha$  and demanding that the off-diagonal terms vanish, we obtain a relation among  $\theta$ ,  $\ell_1$  and  $\ell_2$ , namely

$$\ell_1 \cosh 2\theta - \ell_2 \sinh 2\theta = \frac{1}{2}\epsilon(k) \quad (42)$$

$$\ell_1 \sinh 2\theta - \ell_2 \cosh 2\theta = 0, \quad (43)$$

where  $\epsilon(k)$  is the dispersion depending on the magnitude  $k$  of the quasi-particle momentum  $\mathbf{k}$  due to the rotational symmetry. Solving for the hyperbolic functions in terms of  $\ell_1$ ,  $\ell_2$  and  $\epsilon(k)$ , and using the identity  $\cosh^2 2\theta - \sinh^2 2\theta = 1$ , yields an expression of  $\epsilon(k)$  in terms of  $\ell_1$  and  $\ell_2$ , which in turn relates it to the  $g$ ,

$$\begin{aligned} \epsilon(k) &= 2(\ell_1^2 - \ell_2^2)^{1/2} \\ &= \left( \left( \frac{k^2}{2m} + \frac{g}{2e} \right)^2 - \left( \frac{g}{e} \right)^2 \left( \frac{e}{2} - 1 \right)^2 \right)^{1/2}, \end{aligned} \quad (44)$$

and the Hamiltonian is

$$H = \sum_{\mathbf{k} \neq 0} \epsilon(k) n(k), \quad (45)$$

where  $n(k) = \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}}$  is the occupation number of the quasi-particle state with energy  $\epsilon$  and momentum  $\mathbf{k}$ , depending on the magnitude of  $\mathbf{k}$  again thanks to the rotational symmetry.

## Kolmogorov Scaling

From our locally scale invariant model we have seen that in order to describe the superfluid state a quasiparticle with energy that scales linearly with momentum in a certain range of momentum values emerges. Unlike the standard Bogoliubov quasiparticle result the second sound value is not fixed by the theory but has to be taken from experiment but the linear relationship between energy and momentum is present in both approaches in the small momentum region. This result was obtained, as in the Bogoliubov approach, by putting in details of the superfluid helium state in terms of a condensate. For this calculation to be valid the interaction between quasiparticles must be small. This is essential for the idea of quasiparticle to be useful. These two features, both present in our effective model, allow us to use the method of weak wave turbulence to determine the turbulent properties of superfluid helium. Essentially this means determining the way energy for momentum  $k$  scale with momentum and to check if the scaling exponent of energy calculated agrees with the observed Kolmogorov exponent.

We have already found the scaling exponent for quasiparticles. We now need to find if the occupation number for momentum  $k$  scales with  $k$  and to determine this exponent. The key calculation to do this, in weak turbulence, involves setting up a Boltzmann type of equation for the occupation number of quasiparticles with a given momentum and checking to see if it has a time independent scaling solution *i.e.* a solution where the occupation number scales with the momentum exists. The procedure outlined is a standard step of weak wave turbulence. The calculation for a quartic interaction term has been done and we can simply use the known results to write down scaling exponent for occupation number for our case. Once this exponent is determined the Kolmogorov exponent is fixed.

For our quasiparticle system with a weak quartic interaction term the Boltzmann time evolution equation for the quasiparticle excitation number of momentum  $k$  is obtained from the system Hamiltonian. Time independent solutions to this evolution equation with a scaling law behaviour have energy [14–17]

$$\begin{aligned} E(k) &= n(k)\epsilon(k) \\ &\sim k^{-\gamma/3}, \end{aligned} \tag{46}$$

where the exponent  $\gamma$  is expressed as  $\gamma = \frac{3d+2\beta}{\alpha} - 4$  in terms of the spatial dimension  $d$  and the exponents of scaling of the coefficient of the quartic term and the energy dispersion, namely

$$T(k) \sim k^\beta \tag{47}$$

$$\epsilon(k) \sim k^\alpha. \tag{48}$$

We have so far discussed the terms quadratic in the raising and lowering operators in the Hamiltonian. The coefficient of the quartic term, which goes as  $1/N$  in the large  $N$  limit that we are considering, is independent of  $k$ , leading to  $\beta = 0$ . As can be seen from (44) if the momenta are

in the range

$$\begin{aligned} \frac{k^2}{2m} &< \frac{g}{e} \\ \frac{k^2}{2m} \left( \frac{k^2}{2m} + \frac{g}{e} \right) &> \frac{g}{e} \left( e - \frac{e^2}{4} - \frac{3}{4} \right) \\ &= 0.12 \left( \frac{g}{e} \right)^2, \end{aligned} \quad (49)$$

then the dispersion is linear in momentum and thus gives  $\alpha = 1$ . Hence,  $\gamma = 5$ , leading, according to (46), to the Kolmogorov scaling law,  $E(k) \sim k^{-5/3}$ , within this range of momentum. Thus, in an appropriate range of momentum we obtain linear dispersion relation and thus weak turbulence and Kolmogorov scaling law from the four wave resonance.

## 4.2 Filaments

In section 3 we obtained field configurations with currents along curves describing filaments. Correlation between two filaments is then understood as the correlation between current supported on a pair of curves, say,  $C_1$  and  $C_2$ , as we shall discuss in this section. The modulus of the condensate  $\psi$  vanishes on the filaments and assumes a non-zero constant value outside the filaments, as described by (20). Adding source terms to the action for the filaments  $C_a$ ,  $a = 1, 2, \dots$ , and setting  $\psi$  to zero the action becomes

$$S_{\text{fil}} = \gamma \int_{\mathbf{R}^3} d^3x \epsilon^{ijk} A_i \partial_j A_k + \sum_a \int_{\mathbf{R}^3} d^3x A_i J_{C_a}^i. \quad (50)$$

Introducing the observables  $W_a = \exp \left( i \int d^3x A_i J^i(\mathbf{x}) \right)$ , the correlator  $\langle W_1 W_2 \rangle$  is obtained by integrating out the gauge field as

$$\langle W_1 W_2 \rangle = \exp \left( \frac{i}{2\gamma} \int dt \int d^3x \int d^3y J_{C_1}^i(\mathbf{x}) G_{ij}(\mathbf{x} - \mathbf{y}) J_{C_2}^j(\mathbf{y}) \right), \quad (51)$$

where  $G_{ij}(\mathbf{x} - \mathbf{y})$  denotes the Green's function associated to the Chern-Simons term, given by

$$L_{ij} G_{jk}(\mathbf{x} - \mathbf{y}) = \delta_{ik} \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (52)$$

where we defined, for compactness of notation, the operator  $L_{ij} = -\epsilon^{ijk} \partial_k$ . Operating with  $L$  from the left on both sides we obtain

$$(L^2)_{ij} G_{jk} = L_{ik} \delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (53)$$

so that the Green's function is given by

$$G_{ij} = L_{kj} (L^{-2})_{ki} \delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (54)$$

Now using  $(L^2)_{ij} = \partial_i \partial_j - \delta_{ij} \partial^2$ , we have

$$(L^2)_{ij} \frac{1}{|\mathbf{x} - \mathbf{y}|} = \partial_i \partial_j \frac{1}{|\mathbf{x} - \mathbf{y}|} - \delta_{ij} \delta(|\mathbf{x} - \mathbf{y}|), \quad (55)$$

leading to

$$(L^{-2})_{ij} \delta^{(3)}(\mathbf{x} - \mathbf{y}) = (L^{-2})_{ik} \partial_k \partial_j \frac{1}{|\mathbf{x} - \mathbf{y}|} - \delta_{ij} \frac{1}{|\mathbf{x} - \mathbf{y}|}. \quad (56)$$

Using this formula in (54) we obtain

$$G_{ij} = -L_{jk} \partial_k \partial_i \frac{1}{|\mathbf{x} - \mathbf{y}|} - L_{ij} \frac{1}{|\mathbf{x} - \mathbf{y}|}. \quad (57)$$

The first term vanishes as  $L_{jk} \partial_k \partial_i = \epsilon^{jkl} \partial_l \partial_k \partial_i = 0$  leaving us with the expression for the Green's function

$$\begin{aligned} G_{ij}(\mathbf{x} - \mathbf{y}) &= \epsilon^{ijk} \partial_k \frac{1}{|\mathbf{x} - \mathbf{y}|} \\ &= -\epsilon^{ijk} \frac{(\mathbf{x} - \mathbf{y})_k}{|\mathbf{x} - \mathbf{y}|^3}. \end{aligned} \quad (58)$$

Inserting this expression for the Green's function in (51) the correlator assumes the form [18–20]

$$\langle W_1 W_2 \rangle = \exp \left( \frac{i}{2\gamma} \int dt \int_{C_1} ds_1 \int_{C_2} ds_2 \epsilon_{ijk} \frac{(\mathbf{x} - \mathbf{y})^k}{|\mathbf{x} - \mathbf{y}|^3} J^i(\mathbf{x}(s_1)) J^j(\mathbf{y}(s_2)) \right), \quad (59)$$

where we have introduced affine variables  $s_1$  and  $s_2$  parametrizing the curves  $C_1$  and  $C_2$  respectively. The interaction energy for a pair of filaments is, then,

$$\int_{C_1} ds_1 \mathbf{B}(s_1) \cdot \mathbf{J}(\mathbf{x}(s_1)), \quad (60)$$

where the field

$$\mathbf{B}(s_1) = \frac{1}{2\gamma} \int_{C_2} ds_2 \frac{\mathbf{J}(\mathbf{y}(s_2)) \times (\mathbf{x}(s_1) - \mathbf{y}(s_2))}{|\mathbf{x}(s_1) - \mathbf{y}(s_2)|^3}$$

is analogous to a magnetic field generated by a current  $\mathbf{J}$ . The equation for a current element of unit mass is therefore

$$\frac{d\mathbf{u}}{dt} = \mathbf{u} \times \mathbf{B}, \quad (61)$$

where  $\mathbf{u} = d\mathbf{x}/dt$ . Recalling that the current is a one-form supported on the filament,  $J(\mathbf{x}) = J_{C_i} dx^i$ , we then derive the equation of a point on one single filament as

$$\frac{d\mathbf{x}}{dt} \sim \mathbf{x} \times \mathbf{v}, \quad (62)$$

with velocity

$$\mathbf{v} = \int \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} \times d\mathbf{r}, \quad (63)$$

We thus have a Biot-Savart type interaction between filaments. Let us emphasize again that the interaction term which leads to the dynamics of filament interactions is not put in by hand in our approach but appears naturally from the requirement of local spatial scale invariance. The velocity

of separation between two colliding filaments within a short time  $\Delta t$  after the collision can be estimated from this. Moreover, the force  $F$  between two small segments  $d\mathbf{x}$  and  $d\mathbf{y}$  on the two curves separated by a small distance  $\mathbf{r}$  has magnitude

$$F = \frac{1}{r^3} d\mathbf{x} \times d\mathbf{y} \cdot \mathbf{r}, \quad (64)$$

where  $r = |\mathbf{r}|$ . The impulse  $K = F\Delta t$  within a short interval of time  $\Delta t$  after the collision is given as

$$K \approx \frac{1}{(\Delta v)^2 / \Delta t}, \quad (65)$$

where  $\Delta v$  is the separation velocity of the filaments in this time. Assuming  $K$  to be a constant for this short time we conclude that

$$(\Delta v)^2 \propto 1/\Delta t, \quad (66)$$

in agreement with experimental observations [5].

## 5 Conclusions

Motivated by the experimental evidence of Kolmogorov scaling in turbulent superfluid flow we have constructed a theory invariant under local spatial scaling which can describe Bose-Einstein condensation. Constructing such a theory is in its own right an interesting theoretical problem as the scale invariance necessitates the introduction of a gauge field and a metric with a Ricci term added to the action. Appearance of the Ricci scalar and the gauge field in a specific combination is crucial for gauge invariance. This procedure has been called Ricci gauging [12]. A three-dimensional Chern Simons term for the gauge field is also allowed. Since the scale invariance prohibits any other term the action thus constructed is unique.

Introducing currents along curves corresponding to the heating of the superfluid then makes the Schroedinger field into a condensate vanishing at the location of the filaments when the density of filaments is low and with constant modulus in the bulk. Such an identification of filament excitations as the zeros of the GP wave functions is a standard assumption but here the identification is not a mathematical ansatz but follows from the result established that filament locations are zeros of the Schroedinger condensate. It is thus a dynamical result of the locally scale invariant theory constructed.

We have studied the excitations of the Schroedinger field by considering fluctuations around the condensate, now interpreted as the Bose-Einstein condensate in a superfluid. Integrating out the gauge field yields a GP-like system as an effective theory for the superfluid excitations, leading to the experimentally tested  $5/3$  Kolmogorov scaling with the observed value for the Kolmogorov exponent.

We then consider the effective theory for interaction between filaments. We only need to consider an action with a number of filaments interacting through the Chern Simons term thanks to the vanishing of the condensate along the filaments. We show that this yields a Biot-Savart type

of interaction between filaments described by currents. This is again the standard way that filaments are regarded and their interactions modelled. Usually the interaction term is introduced as an ansatz. Here it is a consequence of three-dimensional local scale invariance.

We have also shown how the velocity of separation between two filaments after collision depended on the time of separation. This is done using an impulse approximation between filaments based on the Biot-Savart interaction. The result obtained is in agreement with observations [5].

Let us point out that time evolution breaks scale invariance in the effective theory constructed as the standard kinetic energy term scales differently from the other terms if the spatial and temporal coordinates are scaled similarly. As filaments with length scales appear in turbulent superfluid flows this breakdown is acceptable. On the other hand, it is well known that turbulent flows that represent far from equilibrium dynamically generated stationary configurations [15] can have scale invariance for its energy distribution spectrum even though the starting space-time dynamics is not scale invariant.

We thus conclude that an effective theory based on Weyl’s original idea of gauge invariance as local scale invariance is compatible with the existing descriptions used to understand of superfluid turbulence including the interaction dynamics of excitations. Local scale invariance leads to correctly identifying the degrees of freedom and leads to the dynamics of the excitations in the superfluid turbulent phase. It is satisfying that the effective theory links filament locations, postulated to be filament currents which couple to scale gauge fields, with the zeros of the GP-like equation. The approach described to construct locally scale invariant systems is also of theoretical interest as it is a very general method for constructing locally scale invariant effective theories.

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